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Measured Correlations and the Wigner Distribution

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§1 Introduction.

Our aim is to find out how to describe a stochastic dynamical system when the underlying dynamics is quantum mechanical. The standard description in the classical case is by a hierarchy of joint probability distributions $W(a_1)$, $W(a_1,a_2), W(a_1,a_2,a_3),\ldots$ of a dynamical variable $X(t)$ which depends on the time; more precisely, $X(t_1), X(t_2),\ldots$ are random variables and $\int_{M_1} W(a_1)\,da_1$ is the probability that $X(t)$ has values in the interval M_1, $\iint_{M_1 M_2} W(a_1,a_2)\,da_1 da_2$ is the probability that both $X(t_1)$ has values in M_1 and $X(t_2)$ has values in M_2, and so on. These probabilities are estimated from measurements of $X(t_1), X(t_2),\ldots$ and the values obtained are compared with those calculated from a mathematical model of the process. We are not forced to distinguish in principle between the measured joint probabilities and an underlying set of joint probabilities because in those situations which we describe as classical it is possible to separate the measuring process completely from the probability distributions. When the dynamics is quantum mechanical the situation is very different. The received doctrine is that the measurement process disturbs the system so radically that no joint probability distributions in the ordinary sense exist for non-commuting observables. Nevertheless it is found convenient for some purposes to make use of the Wigner distribution function [1] and its generalisations despite the fact that such functions can take on negative values (see Urbanik [2] for a very thorough discussion). Some authors (see section §5) have followed Husimi [3] and smoothed the Wigner function with a Gaussian to ensure positivity, even though this procedure seems somewhat arbitrary.

The fact is that there is a reluctance to consider the representation of the measurement process, yet without such a device it is difficult to discuss the properties of measured probability distributions. One reason for this reluctance is that in the von Neumann description [4] it is possible to deal directly only with discrete observables (that is, observables whose spectrum consists entirely of isolated eigenvalues). Observables having continuous spectrum have first to be

approximated by discrete observables. This is messy and not entirely convincing. In this paper we make use of a class of representations of measurements of restricted accuracy which can be used with an arbitrary observable and which includes as a special case one which reduces to the usual one for a discrete observable. This is in the spirit of the work of Mielnik [5] and Davies and Lewis [6]. Using this class of representations we show that it is possible to conceive of measured joint distributions which take account of the order in which measurements are made and which are always positive, even when the observables do not commute. Furthermore, these measured distributions can be represented as the convolution of a Wigner-type joint distribution function with a function which depends on the measuring process and on the commutators of the observables which are measured. We discuss the relationship with the Husimi transformation in §5.

In order to clarify matters we consider first the representation of the measurement process in the classical situation. In order to avoid unnecessary complications we shall assume that although the measurements are made in a specified order the elapse of time between them can be neglected.

§2 Classical Measurements

We suppose that the state of the system is described by a positive function $\rho(P)$ on phase space Γ which is normalizable, in that $\int_\Gamma \rho(P)\,d\Gamma$ is finite, but we find it convenient to avoid assuming that it is already normalized. The observables are real-valued functions $A_1(P), A_2(P), \ldots$ on Γ. The probability that A_1 takes values in the interval M_1 is given by

$$\int_\Gamma \chi_{M_1}(A_1(P))\,\rho(P)\,d\Gamma \Big/ \int_\Gamma \rho(P)\,d\Gamma$$

where χ_M is the indicator function of the set M:

$$\chi_M(a) = 1 \quad \text{if } a \text{ belongs to } M,$$
$$= 0 \quad \text{otherwise.}$$

We can look upon this probability formula as follows:
We think of the function $\rho(P)$ as describing a distribution of copies of the system in phase space, and the measurement process as a procedure which selects those copies for which A_1 takes values in M_1 and rejects the others thus producing a new state after the measurement described by the function $\chi_{M_1}(A_1(P))\rho(P)$. The probability that A_1 takes values in M_1 is thought of as the fraction of the total number of copies of the system for which A_1 takes values in M_1 and is thus given by $\int_\Gamma \chi_{M_1}(A_1(P))\rho(P)d\Gamma / \int_\Gamma \rho(P)\,d\Gamma$. This interpretation suggests how we might represent measurements having restricted accuracy. The selection procedure is now not completely efficient; it might be represented by $\chi_{M_1}(a)$ convolved with a positive function which, for our future convenience, we write as $|\alpha_1|^2(a)$. In our applications it will be an approximation to a delta-function. The state following the measurement is given by

$$(\chi_{M_1} * |\alpha_1|^2)(A_1(P))\,\rho(P)$$

where

$$(\chi_M * |\alpha|^2)(a) = \int_{-\infty}^{\infty} \chi_M(b)\,|\alpha|^2(a-b)\,db .$$

The restricted accuracy of the measurement is described by the function $|\alpha|^2(a)$ which we assume satisfies

$$\int_{-\infty}^{\infty} |\alpha|^2(a)\,da = 1 \quad , \quad \int_{-\infty}^{\infty} a\,|\alpha|^2(a)\,da = 0 .$$

The measured probability that A_1 takes values in M_1 is thus

$$\int_\Gamma (\chi_M * |\alpha_1|^2)(A_1(P))\,\rho(P)\,d\Gamma \Big/ \int_\Gamma \rho(P)\,d\Gamma .$$

Suppose now that a second measurement is made; this time we select those copies of the system for which A_2 takes values in M_2 and the restricted accuracy is described by the function $|\alpha_2|^2(a)$. The state after this second measurement is represented by

$$(\chi_{M_2} * |\alpha_2|^2)(A_2(P))\,(\chi_{M_1} * |\alpha_1|^2)(A_1(P))\,\rho(P)$$

and the measured probability is given by

$$\int_\Gamma (\chi_{M_2} * |\alpha_2|^2)(A_2(P))\,(\chi_{M_1} * |\alpha_1|^2)(A_1(P))\,\rho(P)\,d\Gamma \Big/ \int_\Gamma \rho(P)\,d\Gamma .$$

We can write this as $\iint_{M_1 \times M_2} W(a_1,a_2)\,da_1 da_2$ where $W(a_1,a_2)$ is the measured joint probability distribution and is given by

$$W(a_1,a_2) = \int_\Gamma |\alpha_2|^2(A_2(P)-a_2)\,|\alpha_1|^2(A_1(P)-a_1)\,\rho(P)\,d\Gamma \Big/ \int_\Gamma \rho(P)\,d\Gamma .$$

In an obvious way we can extend this to an n -fold measured joint probability distribution $W(a_1, a_2, \ldots, a_n)$ of $A_1, \ldots, A_n$ with accuracy described by $|\alpha_1|^2, \ldots, |\alpha_n|^2$.

The separation of the measurement process from the underlying distribution is most conveniently seen by using characteristic functions. Let $\Xi(k_1, \ldots, k_n)$ be the characteristic function of the measured distribution $W(a_1, \ldots, a_n)$:

$$\Xi(k_1, \ldots, k_n) = \int_{\mathbb{R}^n} e^{i(k_1 a_1 + \cdots + k_n a_n)} W(a_1, \ldots, a_n)\, da_1 \ldots da_n$$

and let $\Xi_0(k_1, \ldots, k_n)$ be the characteristic function of the underlying distribution :

$$\Xi_0(k_1, \ldots, k_n) = \int_\Gamma e^{i(k_1 A_1(P) + \cdots + k_n A_n(P))} \rho(P)\, d\Gamma \Big/ \int_\Gamma \rho(P)\, d\Gamma$$

Then an elementary calculation shows that

$$\Xi(k_1, \ldots, k_n) = G(k_1, \ldots, k_n)\, \Xi_0(k_1, \ldots, k_n)$$

where

$$G(k_1, \ldots, k_n) = \widetilde{|\alpha_1|^2}(k_1)\, \widetilde{|\alpha_2|^2}(k_2) \cdots \widetilde{|\alpha_n|^2}(k_n)$$

and

$$\widetilde{|\alpha|^2}(k) = \int_{-\infty}^{\infty} e^{ika} |\alpha|^2(a)\, da \quad .$$

In the case where the $|\alpha_i|^2$ are Gaussian ,

$$|\alpha_i|^2(a_i) = (2\pi\sigma_i^2)^{-1/2} \exp(-a_i^2/2\sigma_i^2) \quad ,$$

we have

$$G(k_1, \ldots, k_n) = \exp\left(-\frac{1}{2}(\sigma_1^2 k_1^2 + \cdots + \sigma_n^2 k_n^2)\right) ,$$

and

$$W(a_1, \ldots, a_n) = \int_{\mathbb{R}^n} \hat{G}(a_1 - b_1, \ldots, a_n - b_n) W(b_1, \ldots, b_n)\, db_1 \cdots db_n$$

where $W(b_1, \ldots, b_n)$ is the underlying joint distribution given by

$$W(b_1, \ldots, b_n) = \hat{\xi}_0(b_1, \ldots, b_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(k_1 b_1 + \cdots + k_n b_n)} \xi_0(k_1, \ldots, k_n)\, dk_1 \cdots dk_n .$$

Notice that G depends only on the measuring process as described by the function α. The closer α approximates a delta function the closer the measured distribution $\hat{W}$ approximates the underlying distribution W.

§3 Quantum Mechanical Measurements

To avoid misunderstanding we emphasise that when we speak of 'measurement' we have in mind 'measurement with selection' (what Pauli [7] called 'measurement of the second kind'), so that the measurement changes the state of the system. We do not enquire here how this may come about although this is an interesting and deep question. We are only concerned with the consequences of assuming a particular representation for the change of state brought about by measurement. A state is described by a positive operator ρ on a Hilbert space $\mathcal{H}$ which is normalizable in the sense that trace ρ is finite, but again we find it convenient to avoid assuming that it is already normalized. Observables are self-adjoint operators on $\mathcal{H}$.

Let us recall how the effect of the measurement of a discrete observable A is represented. Let $x_1, x_2, \ldots$ be the eigenvalues of A and let $P_1, P_2, \ldots$ be the associated projection operators so that

$$A = \sum_i x_i P_i \quad .$$

It is commonly accepted that a measurement which selects those copies of the system having values of A in M in the state ρ changes the state into a new one described by

$$\sum_{\{i : x_i \in M\}} P_i \rho P_i \quad .$$

This representation can be derived from a variety of assumptions about the optimal character of the measurement (such that the disturbance of the state is in some sense minimal; see Luders [8], Goldberger and Watson [9], Furry [10], Davies and Lewis [6]). It should be noted that in the theory of angular correlations a representation

$$\sum_{\{i : x_i \in M\}} e_i P_i \rho P_i$$

is used where the parameters, which can take values between 0 and 1, describe the efficiency of the counter (see Coester and Jauch [11] and references given there).

For observables having continuous spectrum there is no such optimal measurement: the continuous spectrum can only be measured approximately, and so we look for guidance to the classical case. Again we will use a function α to describe the accuracy of the measurement; α can be complex-valued but we assume that

$$\int_{-\infty}^{\infty} |\alpha|^2(a)\, da = 1, \quad \int_{-\infty}^{\infty} a\,|\alpha|^2(a)\, da = 0,$$

and putting $\sigma^2 = \int_{-\infty}^{\infty} a^2 |\alpha|^2(a)\, da$ we call σ the width of the instrument. In the classical case we could write the state after the measurement as

$$\int_M \alpha(A(P)-a)\, \rho(P)\, \overline{\alpha(A(P)-a)}\, da .$$

In the quantum-mechanical case we imitate this and write

$$\rho' = \int_M \alpha(A - a1)\, \rho\, \alpha(A-a1)^*\, da$$

where we use the spectral representation $A = \int_{-\infty}^{\infty} \lambda\, dE_\lambda$ to give meaning to $\alpha(A-a1)$:

$$\alpha(A-a1) = \int_{-\infty}^{\infty} \alpha(\lambda - a)\, dE_\lambda .$$

It follows that ρ' is positive and has finite trace. The particular case in which, for some $\epsilon > 0$, the function α is given by

$$\alpha(a) = \epsilon^{-1/2} \chi_{[-\epsilon/2,\, \epsilon/2]}(a)$$

is interesting when the observable A is discrete and the minimum distance between its eigenvalues is greater than ϵ.

Then

$$\alpha(A-a1) = \int_{-\infty}^{\infty} \epsilon^{-1/2} \chi_{[-\epsilon/2,\,\epsilon/2]}(\lambda-a)\, dE_\lambda$$

$$= \epsilon^{-1/2} P_i \quad , \; a \in [x_i - \epsilon/2,\, x_i + \epsilon/2]$$

$$= 0 \qquad \text{otherwise,}$$

so that

$$\alpha(A-a1)\rho\alpha(A-a1)^* = \sum_i \epsilon^{-1} \chi_{[x_i-\epsilon/2,\,x_i+\epsilon/2]}(a)\; P_i \rho P_i$$

and

$$\int_M \alpha(A-a1)\rho\alpha(A-a1)^* da = \sum_{\{i;\, x_i \in M\}} e_i P_i \rho P_i$$

where $e_i = l_i/\epsilon_i$ and l_i is the length of the interval

$$M \cap [x_i - \epsilon/2,\, x_i + \epsilon/2] .$$

§4 Measured Joint Distributions in Quantum Theory

In this section we investigate the properties of measured joint distributions defined by means of the representation of the change of state discussed in the previous section. In order to avoid the use of lengthy phrases we introduce some unconventional notation. We put

$$\mathcal{E}_A^{\alpha}(M)\rho = \int_M \alpha(A-a1)\rho\,\alpha(A-a1)^* da .$$

for the state conditioned by the measurement of A in the interval M with accuracy described by α . Then the probability of getting a result in M in such a measurement is

$$\mathbb{P}_A(M) = \mathrm{trace}(\mathcal{E}_A^{\alpha}(M)\rho)/\mathrm{trace}(\rho) .$$

For repeated measurements we have

$$\mathbb{P}_{A_1A_2}(M_1\times M_2) = \mathrm{trace}(\mathcal{E}_{A_2}^{\alpha_2}(M_2)\,\mathcal{E}_{A_1}^{\alpha_1}(M_1)\rho)/\mathrm{trace}(\rho),$$

$$\mathbb{P}_{A_1A_2A_3}(M_1\times M_2\times M_3) = \mathrm{trace}(\mathcal{E}_{A_3}^{\alpha_3}(M_3)\,\mathcal{E}_{A_2}^{\alpha_2}(M_2)\,\mathcal{E}_{A_1}^{\alpha_1}(M_1)\rho)/\mathrm{trace}\,\rho ,$$

$$\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

and so on. A straight-forward calculation shows that

$$\mathbb{P}_{A_1}(M_1) = \int_{M_1} W(a_1)\,da_1 ,$$

$$\mathbb{P}_{A_1A_2}(M_1\times M_2) = \iint_{M_1\times M_2} W(a_1,a_2)\,da_1 da_2 ,$$

$$\mathbb{P}_{A_1A_2A_3}(M_1\times M_2\times M_3) = \iiint_{M_1\times M_2\times M_3} W(a_1,a_2,a_3)\,da_1 da_2 da_3 ,$$

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where

$$W_{A_1}(a_1) = \text{trace}\left(\alpha_1(A_1 - a_1 1)\,\rho\,\alpha_1(A_1 - a_1 1)^*\right)/\text{trace}(\rho)\,,$$

$$W_{A_1A_2}(a_1,a_2) = \text{trace}\left(\alpha_2(A_2 - a_2 1)\alpha_1(A_1 - a_1 1)\,\rho\,\alpha_1(A_1 - a_1 1)^*\alpha_2(A_2 - a_2 1)^*\right)/\text{trace}(\rho),$$

. . . . . .

and so on. The notation emphasises that the joint probability distribution $W_{A_1 \cdots A_n}(a_1, \cdots, a_n)$ depends on the order in which the measurements are made. It is easy to check that they satisfy :

(1) $W_{A_1 \cdots A_n}(a_1, \cdots, a_n) \geqslant 0$.

(2) $\int_{-\infty}^{\infty} W_{A_1 \cdots A_{n-1}A_n}(a_1, \ldots, a_{n-1}, a_n)\, da_n = W_{A_1 \cdots A_{n-1}}(a_1, \ldots, a_{n-1})$.

(3) In general $W_{A_1 \cdots A_n}(a_1, \ldots, a_n) \neq W_{A_{\pi(1)} \cdots A_{\pi(n)}}(a_{\pi(1)}, \ldots, a_{\pi(n)})$

for a permutation π of $1, \ldots, n$ unless, of course, the $A_1, \ldots, A_n$ commute.

It follows from the requirement $\int_{-\infty}^{\infty} a\,|\alpha|^2(a)\,da = 0$ that

$$\int_{-\infty}^{\infty} a\, W_A(a)\, da = \text{trace}(\rho A)/\text{trace}(\rho)\,.$$

In contrast to properties (1) and (3), the obvious generalisation $W(a_1, \ldots, a_n)$ of the Wigner distribution function ,

$$W(a_1, \ldots, a_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(k_1 a_1 + \cdots + k_n a_n)}\, \xi_0(k_1, \ldots, k_n)\, dk_1 \cdots dk_n$$

where

$$\xi_0(k_1,\dots,k_n) = \operatorname{trace}\left(\rho\, e^{i(k_1A_1+\cdots+k_nA_n)}\right)/\operatorname{trace}(\rho),$$

can take on negative values and is independent of the order of the observables

The main result of this paper is the somewhat surprising result that if the commutators of the A's are c-numbers we have a factorization of the characteristic function $\xi_{A_1\cdots A_n}(k_1,\dots,k_n)$ of the joint distribution $W_{A_1\cdots A_n}(a_1,\dots,a_n)$:

$$\xi_{A_1\cdots A_n}(k_1,\dots,k_n) = G_{A_1\cdots A_n}(k_1,\dots,k_n)\,\xi_0(k_1,\dots,k_n)$$

as in the classical case, where

$$\xi_{A_1\cdots A_n}(k_1,\dots,k_n) = \int_{\mathbb{R}^n} e^{i(k_1a_1+\cdots+k_na_n)} W_{A_1\cdots A_n}(a_1,\dots,a_n)\,da_1\cdots da_n.$$

In the quantum case, however, $G_{A_1\cdots A_n}$ depends on the commutators of the A_i as well as on the functions α_i. More precisely, let $A_1,\dots,A_n$ be self-adjoint operators such that

$$[A_i, A_j] = i\,c_{ij}\,1 \quad , \quad (i,j=1,\dots,n)$$

then

$$\xi_{A_1\cdots A_n}(k_1,\dots,k_n) = G_{A_1\cdots A_n}(k_1,\dots,k_n)\,\xi_0(k_1,\dots,k_n),$$

where

$$G_{A_1}(k_1) = \widetilde{|\alpha_1|^2}(k_1),$$

and

$$G_{A_1\cdots A_n}(k_1,\dots,k_n) = G_{A_2\cdots A_n}(k_2,\dots,k_n)\int_{-\infty}^{\infty}\check{\alpha}_1(k)\,\overline{\check{\alpha}}_1(k_1-k)\cos\left\{\tfrac{1}{2}(k_1-2k)\sum_{j=2}^{n}k_j c_{1j}\right\}\frac{dk}{2\pi}$$

The formula for the $G_{A_1 \cdots A_n}$ can be computed explicitly when the α_i are Gaussian. Putting $|\alpha_i|^2(a) = (2\pi\sigma_i^2)^{-1/2} \exp\{-a^2/2\sigma_i^2\}$ we have

$$G_{A_1 \cdots A_n}(k_1, \ldots k_n) = \exp\{-\tfrac{1}{2}(\sigma_1^2 k_1^2 + \cdots + \sigma_n^2 k_n^2)\} \exp\left\{-\frac{1}{2\sigma_1^2}\left(\sum_{j=2}^{n} c_{1j} k_j/2\right)^2\right\} \cdots$$

$$\cdots \exp\left\{-\frac{1}{2\sigma_{n-1}^2}(c_{n-1\,n} k_n/2)^2\right\}.$$

The first term on the right hand side is the classical expression, the second term expresses the change in the distribution due to the measurement of A_1, the third term expresses the change in the distribution due to the measurement of A_2, and so on. We give the proof in §6 but first discuss the result in more detail.

§5 The Uncertainty Principle and the Husimi Transformation

The proof of the uncertainty principle as given in most deductive treatments of quantum theory goes back to Pauli (see Weyl [12]). What it proves is that in a given state the dispersion Δp is related to the dispersion Δq by the inequality $\Delta q \Delta p \geq \hbar/2$. But it is usually stated as: the measurement of q disturbs a state so much that a subsequent measurement of p cannot have arbitrarily small dispersion. The result of the previous section leads to a proof of the principle in this second form.

Consider the case of Gaussian α_i and put $A_1 = q$, $A_2 = p$ so that $c_{12} = \hbar$. Then

$$W_{qp}(a_1, a_2) = \int_{\mathbb{R}^2} \hat{G}_{qp}(a_1 - b_1, a_2 - b_2) W(b_1, b_2)\, db_1 db_2$$

where

$$\hat{G}_{qp}(b_1, b_2) = (2\pi\sigma_1\sigma_2')^{-1} \exp\left\{ - b_1^2/2\sigma_1^2 - b_2^2/2\sigma_2'^2 \right\}$$

and

$$\sigma_2'^2 = \sigma_2^2 + \hbar^2/4\sigma_1^2 .$$

Thus $\sigma_2'^2 > \hbar^2/4\sigma_1^2$ so that $\sigma_2'\sigma_1 > \hbar/2$. The limit $\hbar/2$ is approached as the width σ_2' of the second measurement goes to zero.

It appears that it was Husimi [3] who first pointed out that a suitable smoothing of the Wigner distribution makes it everywhere positive. This idea has been used by McKenna and Frisch [13], rediscovered by Kano [14], and applied to scattering theory by Iagolnitzer [15], who appreciated that it is connected with measurement theory but did not make that connection explicit. We are indebted to

Professor H. Primas for drawing our attention to the Husimi transformation and for supplying references. The Husimi transform $H(a_1, a_2)$ of the Wigner distribution $W(a_1, a_2)$ is given by

$$H(a_1, a_2) = \int_{\mathbb{R}^2} K(a_1 - b_1, a_2 - b_2)\, W(b_1, b_2)\, db_1\, db_2$$

where

$$K(a_1, a_2) = (\pi\hbar)^{-1} \exp\{ - a_1^2/2\Delta^2 - 2\Delta^2 a_2^2/\hbar^2 \} .$$

This is just the limit of $\hat{G}_{qp}(a_1, a_2)$ as $\sigma_2 \to 0$.

§6 Proof of the Main Result

We use an obvious notation and write

$$W(1,\dots,s) = \langle \alpha_1 \alpha_2 \cdots \alpha_s \alpha_s^* \cdots \alpha_1^* \rangle = \operatorname{trace}(\rho\, \alpha_1 \cdots \alpha_s \alpha_s^* \cdots \alpha_1^*)/\operatorname{trace}(\rho)$$

Then $\hat{S}(1,\dots,s) = \langle A(1,\dots,s) \rangle$ where

$$A(1,\dots,s) = \int_{\mathbb{R}^s} e^{i(k_1 A_1 + \cdots + k_s A_s)} \alpha_1(A_1 - a_1) \cdots \alpha_s(A_s - a_s) \alpha_s(A_s - a_s)^* \cdots da_1 \cdots da_s .$$

We assert that

$$A(1,\dots,s) = G(1,\dots,s)\, e^{i(k_1 A_1 + \cdots + k_s A_s)}$$

where $G(1,\dots,s)$ is a c-number. We use induction on s.

Suppose the assertion holds for $1,2,\dots,s$ variables. Then

$$A(1,\dots,s+1) = G(2,\dots,s+1) \int_{-\infty}^{\infty} e^{ik_1 A_1} \alpha_1(A_1 - a_1)\, e^{i\sum_2^{s+1} k_j A_j}\, \alpha_1(A_1 - a_1)^*\, da_1$$

$$= G(2,\dots,s+1) \int_{-\infty}^{\infty} \check{\alpha}_1(k)\, \check{\alpha}_1(k_1 - k)\, e^{ik_1 A_1}\, e^{i\sum_{j=2}^{s+1} k_j A_j}\, e^{i(k_1 - k)A_1} \frac{dk}{2\pi} .$$

Using the Weyl form of the commutation relations

$$e^{ik_i A_i}\, e^{ik_j A_j} = e^{-ik_i k_j C_{ij}/2}\, e^{i(k_i A_i + k_j A_j)} .$$

we have

$$A(1,\dots,s+1) = G(2,\dots,s+1) \int_{-\infty}^{\infty} \check{\alpha}_1(k)\, \check{\alpha}_1(k_1 - k)\, e^{-\frac{i}{2}(k_1 - 2k)\sum_{j=2}^{s+1} k_j C_{1j}}\, e^{i\sum_1^{s+1} k_j A_j} \frac{dk}{2\pi} .$$

Thus

$$G(1,\dots,s+1) = G(2,\dots,s) \int_{-\infty}^{\infty} \check{\alpha}_1(k)\, \check{\alpha}_1(k_1 - k) \cos\left\{\tfrac{1}{2}(k_1 - 2k)\sum_{j=1}^{s} k_j C_{1j}\right\} \frac{dk}{2\pi} .$$

Direct calculation gives $A(s) = |\check{\alpha}_s|^2(k_s)\, e^{ik_s A_s}$ and the proof is complete.

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